

Mean field exponents and small quark masses

A. Höll[†], P. Maris[‡] and C.D. Roberts[‡]

[†]*Fachbereich Physik, Universität Rostock, D-18051 Rostock, Germany*

[‡]*Physics Division, Building 203, Argonne National Laboratory, Argonne Illinois 60439-4843, USA*

(Pacs Numbers: 11.10.Wx, 12.38.Mh, 24.85.+p, 05.70.Fh)

Abstract

We demonstrate that the restoration of chiral symmetry at finite- T in a class of confining Dyson-Schwinger equation (DSE) models of QCD is a mean field transition, and that an accurate determination of the critical exponents using the chiral and thermal susceptibilities requires very small values of the current-quark mass: $\log_{10}(m/m_u) \lesssim -5$. Other classes of DSE models characterised by qualitatively different interactions also exhibit a mean field transition. Incipient in this observation is the suggestion that mean field exponents are a result of the gap equation's fermion substructure and not of the interaction.

It is anticipated that the restoration of chiral symmetry, which accompanies the formation of a quark-gluon plasma at finite temperature, T , is an equilibrium, second-order phase transition. Such transitions are completely characterised by two critical exponents: (β, δ) , which describe the response of any one of the equivalent chiral order parameters, \mathcal{X} , to changes in T and in the current-quark mass, m . Denoting the critical temperature by T_c , and introducing the reduced-temperature $t := T/T_c - 1$ and reduced mass $h := m/T$, then

$$\mathcal{X} \propto (-t)^\beta, \quad t \rightarrow 0^-, \quad h = 0, \quad (1)$$

$$\mathcal{X} \propto h^{1/\delta}, \quad h \rightarrow 0^+, \quad t = 0. \quad (2)$$

Calculating the critical exponents is an important contemporary goal because of the widely conjectured notion of *universality*, which states that their values depend only on the symmetries of the theory, the dimension of space, and whether or not the interaction is short-range. In this case many theories could be grouped into a single universality class labelled by (β, δ) , and chiral symmetry restoration in QCD might be describable by a much simpler model.

The success of the nonlinear σ -model in describing long-wavelength pion dynamics underlies a conjecture [1] that the restoration of chiral symmetry at finite- T in 2-flavour QCD is characterised by a free energy whose interaction is just that of the 3-dimensional, $N = 4$ Heisenberg magnet ($O(4)$ model). This theory has been studied extensively and its critical exponents are [2]: $\beta^H = 0.38 \pm 0.01$, $\delta^H = 4.82 \pm 0.05$.

The conjecture that 2-flavour QCD is in the $O(4)$ universality class was reconsidered in Ref. [3], where it is argued that the compositeness of QCD's mesons affects the nature of the phase transition. The Gross-Neveu model, defined by the Euclidean action

$$\int d^{d+1}x \left[\bar{\psi}(i\gamma \cdot \partial + m + g\sigma)\psi - \frac{1}{2}\sigma^2 \right], \quad (3)$$

is presented as a counter-example. Arguments kindred to those of Ref. [1] suggest that the chiral transition in this model should be described by a d -dimensional Ising model because of the discrete $\sigma \rightarrow -\sigma$ symmetry, which is manifest after integrating out the fermions.

At $T = 0$ the Gross-Neveu model exhibits chiral symmetry restoration as the coupling $g \rightarrow g_c^+$ and at leading order in a $1/N_F$ -expansion, where N_F is the number of fermions, the critical exponents are easily calculated by exploiting the divergence that appears as the gap vanishes in the gap equation: $\beta_{T=0}^{\text{GN}} = 1/(d-1)$, $\delta_{T=0}^{\text{GN}} = d$. The same analysis applies at finite- T but in this case the absence of fermion Matsubara-frequency zero-modes ensures that the integrand in the gap equation is finite as the gap vanishes, so that: $\beta_{T \neq 0}^{\text{GN}} = \beta^{\text{MF}} = (1/2)$, $\delta_{T \neq 0}^{\text{GN}} = \delta^{\text{MF}} = 3$. These exponents are apparently unchanged by $1/N_F$ -corrections [3] in which case the finite temperature transition is characterised by mean field exponents, independent of the dimension, *not* Ising model exponents. If this is correct then a description of the transition in terms of elementary meson fields is inadequate in the vicinity of the second-order phase transition [3].

However, in a subsequent study [4] of a Yukawa model, which is equivalent to the Gross-Neveu model in the large- N_F limit, it was argued that the chiral transition *is* described by the expected Ising model exponents but only on a domain of width $\sim 1/N_F$ in the vicinity of the critical-temperature and/or -external-field, with mean field exponents describing the evolution of the order parameters outside that domain. In this case $1/N_F$ -corrections do appear to be important.

These simple examples illustrate why the universality class containing QCD remains uncertain and why it will be difficult to determine if, as is plausible, the QCD chiral transition exhibits similar complexities; e.g., mean field exponents which may, or may not, evolve into $O(4)$ exponents very near the critical-coupling and/or chiral limit. We explore this question by searching for a systematic trend in a comparison of (β, δ) calculated in a range of classes of Dyson-Schwinger equation models of QCD.

Calculating the critical exponents directly from Eqs. (1) and (2) is often difficult because of numerical noise near the critical temperature. Another method is to consider [5] the chiral and thermal susceptibilities:

$$\chi_h(t, h) := \left. \frac{\partial \mathcal{X}(t, h)}{\partial h} \right|_t, \quad \chi_t(t, h) := \left. \frac{\partial \mathcal{X}(t, h)}{\partial t} \right|_h. \quad (4)$$

At each h , $\chi_h(t, h)$ and $\chi_t(t, h)$ are smooth functions of t with maxima at the pseudocritical points t_{pc}^h and t_{pc}^t , where

$$t_{\text{pc}}^h \propto h^{1/(\beta\delta)} \propto t_{\text{pc}}^t. \quad (5)$$

Since $\beta\delta > 0$, the pseudocritical points approach the critical point, $t = 0$, as $h \rightarrow 0^+$ and

$$\chi_h^{\text{pc}} := \chi_h(t_{\text{pc}}^h, h) \propto h^{-z_h}, \quad z_h := 1 - \frac{1}{\delta}, \quad (6)$$

$$\chi_t^{\text{pc}} := \chi_t(t_{\text{pc}}^t, h) \propto h^{-z_t}, \quad z_t := \frac{1}{\beta\delta} (1 - \beta). \quad (7)$$

Therefore, by locating the pseudocritical points and plotting the peak-height of the susceptibilities as a function of h one can determine T_c , β and δ .

This method was employed [6] in numerical simulations of finite- T lattice-QCD on an $8^3 \times 4$ -lattice with three values of the current-quark mass: $a m_q^{\text{lat}} = 0.075, 0.0375, 0.02$, which for $1/a \sim 4 T_c$, $T_c \approx 150 \text{ MeV}$, correspond to $m_q^{\text{lat}} = 45, 23, 12 \text{ MeV}$, and yielded: $\beta^{\text{lat}} = 0.30 \pm 0.08$, $\delta^{\text{lat}} = 4.3 \pm 0.5$. The values are consistent with a straightforward realisation of the $O(4)$ hypothesis, however, the errors are large. The situation deteriorates further when more recent simulations on larger lattices and with smaller quark masses are taken into account [7]. A value of $1/\delta^{\text{lat}} = 0$ is consistent with the new data, and this can be interpreted as a signal of a *first order* transition. Our calculations indicate that this unexpected result may be an artefact, due in part to the large values of the quark mass that contemporary lattice-QCD simulations are restricted to.

Herein we analyse $\chi_h(t, h)$ and $\chi_t(t, h)$ in a class of confining DSE models that underlies many successful phenomenological applications at both zero and finite- (T, μ) [8]. The foundation of our study, as for many others [9–13], is the renormalised quark DSE

$$S^{-1}(p_{\omega_k}) := i\vec{\gamma} \cdot \vec{p} A(p_{\omega_k}) + i\gamma_4 \omega_k C(p_{\omega_k}) + B(p_{\omega_k}) \quad (8)$$

$$= Z_2^A i\vec{\gamma} \cdot \vec{p} + Z_2 (i\gamma_4 \omega_k + m_{\text{bm}}) + \Sigma'(p_{\omega_k}). \quad (9)$$

Here $(p_{\omega_k}) := (\vec{p}, \omega_k)$ with $\omega_k = (2k + 1)\pi T$ the fermion Matsubara frequency, and m_{bm} is the Lagrangian current-quark bare mass. The regularised self energy is

$$\Sigma'(p_{\omega_k}) = i\vec{\gamma} \cdot \vec{p} \Sigma'_A(p_{\omega_k}) + i\gamma_4 \omega_k \Sigma'_C(p_{\omega_k}) + \Sigma'_B(p_{\omega_k}), \quad (10)$$

$$\Sigma'_{\mathcal{F}}(p_{\omega_k}) = \int_{l,q}^{\bar{\Lambda}} \frac{4}{3} g^2 D_{\mu\nu}(\vec{p} - \vec{q}, \omega_k - \omega_l) \frac{1}{4} \text{tr} [\mathcal{P}_{\mathcal{F}} \gamma_\mu S(q_{\omega_l}) \Gamma_\nu(q_{\omega_l}; p_{\omega_k})], \quad (11)$$

where: $\mathcal{F} = A, B, C$; A, B, C are functions only of $|\vec{p}|^2$ and ω_k^2 ; $\mathcal{P}_A := -(Z_1^A/|\vec{p}|^2)i\vec{\gamma} \cdot \vec{p}$, $\mathcal{P}_B := Z_1$, $\mathcal{P}_C := -(Z_1/\omega_k)i\gamma_4$; and $\int_{l,q}^{\bar{\Lambda}} := T \sum_{l=-\infty}^{\infty} \int^{\bar{\Lambda}} d^3q/(2\pi)^3$, with $\int^{\bar{\Lambda}}$ a mnemonic to represent a translationally invariant regularisation of the integral and $\bar{\Lambda}$ the regularisation mass-scale. In renormalising the quark DSE we require

$$S^{-1}(p_{\omega_0}) \Big|_{|\vec{p}|^2 + \omega_0^2 = \zeta^2} = i\vec{\gamma} \cdot \vec{p} + i\gamma_4 \omega_0 + m_R, \quad (12)$$

and hence the renormalised self energies are

$$\mathcal{F}(p_{\omega_k}; \zeta) = \xi_{\mathcal{F}} + \Sigma'_{\mathcal{F}}(p_{\omega_k}; \bar{\Lambda}) - \Sigma'_{\mathcal{F}}(\zeta_{\omega_0}^-; \bar{\Lambda}), \quad (13)$$

$(\zeta_{\omega_0}^-)^2 := \zeta^2 - \omega_0^2$, $\mathcal{F} = A, B, C$, $\xi_A = 1 = \xi_C$, and $\xi_B = m_R(\zeta)$.

Equations (9)-(13) define the exact QCD *gap equation*. It has obvious qualitative similarities to the Gross-Neveu gap equation, in particular the explicit dependence on ω_k .

In Eq. (11), $\Gamma_{\nu}(q_{\omega_l}; p_{\omega_k})$ is the renormalised dressed-quark-gluon vertex whose structure at $T = 0$ has been much considered [14]. It is a connected, irreducible three-point function that should not exhibit light-cone singularities in covariant gauges; i.e., it should be regular at $(\vec{p} - \vec{q})^2 + (\omega_k - \omega_l)^2 = 0$ [15]. A number of *Ansätze* with this property have been proposed and employed, and it has become clear that the judicious use of the rainbow truncation ($\Gamma_{\nu}(q_{\omega_l}; p_{\omega_k}) = \gamma_{\nu}$) in Landau gauge provides phenomenologically reliable results [16]. We employ it herein, in which case a mutually consistent constraint is $Z_1 = Z_2$ and $Z_1^A = Z_2^A$. The rainbow truncation is the leading term in a $1/N_c$ -expansion of $\Gamma_{\nu}(q_{\omega_l}; p_{\omega_k})$.

$D_{\mu\nu}(p_{\Omega_k})$ in Eq. (11) is the renormalised dressed-gluon propagator, which has the form

$$g^2 D_{\mu\nu}(p_{\Omega_k}) = P_{\mu\nu}^L(p_{\Omega_k}) \Delta_F(p_{\Omega_k}) + P_{\mu\nu}^T(p_{\Omega_k}) \Delta_G(p_{\Omega_k}), \quad (14)$$

$$P_{\mu\nu}^T(p_{\Omega_k}) := \begin{cases} 0, & \mu \text{ and/or } \nu = 4, \\ \delta_{ij} - \frac{p_i p_j}{p^2}, & \mu, \nu = i, j = 1, 2, 3, \end{cases} \quad (15)$$

with $P_{\mu\nu}^T(p_{\Omega_k}) + P_{\mu\nu}^L(p_{\Omega_k}) = \delta_{\mu\nu} - p_{\mu} p_{\nu} / \sum_{\alpha=1}^4 p_{\alpha} p_{\alpha}$; $\mu, \nu = 1, \dots, 4$. ($\Omega_k := 2\pi kT$ is the boson Matsubara frequency.) The primary class of DSE models we consider is that in which the long-range part of the interaction, $g^2 D_{\mu\nu}(p_{\Omega_k})$, is an integrable infrared singularity [17]. We write

$$\Delta_F(p_{\Omega_k}) = \mathcal{D}(p_{\Omega_k}; m_g), \quad \Delta_G(p_{\Omega_k}) = \mathcal{D}(p_{\Omega_k}; 0), \quad (16)$$

$$\mathcal{D}(p_{\Omega_k}; m_g) := 2\pi^2 D \frac{2\pi}{T} \delta_{0k} \delta^3(\vec{p}) + \mathcal{D}_M(p_{\Omega_k}; m_g), \quad (17)$$

where D is a mass-scale parameter and $\mathcal{D}_M(p_{\Omega_k}; m_g)$ may be large in the vicinity of $p_{\Omega_k}^2 = 0$ but must be finite. This type of infrared singularity is motivated by $T = 0$ studies of the gluon DSE. Studies in axial gauge [18], where ghost contributions are absent, and in Landau gauge [19], when their contributions are small, indicate that $D_{\mu\nu}(k)$ is significantly enhanced in the vicinity of $k^2 = 0$ relative to a free gauge-boson propagator. In the neighbourhood of $k^2 = 0$ the solution is a regularisation of $1/k^4$ as a distribution, and the enhancement persists to $k^2 \sim 1 \text{ GeV}^2$ where a perturbative analysis becomes quantitatively reliable.

The model of Refs. [11] is obtained with $D := \eta^2/2$, $\mathcal{D}_{M:=A}(p_{\Omega_k}; m_g) \equiv 0$, and the mass-scale $\eta = 1.06 \text{ GeV}$ fixed [17] by fitting π - and ρ -meson masses at $T = 0$. It is an ultraviolet

finite model and hence the renormalisation point and cutoff can be removed simultaneously. A current-quark mass of $m = 12 \text{ MeV}$ yields $m_\pi = 140 \text{ MeV}$. As an infrared-dominant model, it represents the behaviour of $g^2 D_{\mu\nu}(p_{\Omega_k})$ poorly away from $p_{\Omega_k}^2 \approx 0$. However, the artefacts this introduces are easily identified and the model exhibits many features in common with more sophisticated *Ansätze*.

The model of Ref. [12] is obtained with $D := (8/9) m_t^2$ and

$$\mathcal{D}_{\text{M:=B}}(p_{\Omega_k}; m_g) = \frac{16}{9} \pi^2 \frac{1 - e^{-s_{\Omega_k}/(4m_t^2)}}{s_{\Omega_k}}, \quad (18)$$

$s_{\Omega_k} := p_{\Omega_k}^2 + m_g^2$, where $m_g^2 = (8/3) \pi^2 T^2$. The mass-scale $m_t = 0.69 \text{ GeV} = 1/0.29 \text{ fm}$ marks the boundary between the perturbative and nonperturbative domains, and was also fixed [20] by requiring a good description of π - and ρ -meson properties at $T = 0$. At a renormalisation point of $\zeta = 9.47 \text{ GeV}$, $m_R = 1.1 \text{ MeV}$ yields $m_\pi = 140 \text{ MeV}$. This model adds a Coulomb-like short-range interaction to that of Refs. [11], thereby improving its ultraviolet behaviour. The ratio of the coefficients in the two terms of Eq. (17) is chosen so that the long-range effects associated with $\delta_{0k} \delta^3(\vec{p})$ are completely cancelled at short-distances [20].

We also consider a finite- T extension of Ref. [16] defined by

$$\mathcal{D}_{\text{M:=C}}(p_{\Omega_k}; m_g) = \frac{4\pi^2}{\omega^6} D s_{\Omega_k} e^{-s_{\Omega_k}/\omega^2} + \frac{8\pi^2 \gamma_m}{\ln \left[\tau + \left(1 + s_{\Omega_k}/\Lambda_{\text{QCD}}^2 \right)^2 \right]} \frac{1 - e^{-s_{\Omega_k}/(4m_t^2)}}{s_{\Omega_k}}, \quad (19)$$

with $\tau = e^2 - 1$, $\gamma_m = 12/25$, $m_g^2 = (16/5) \pi^2 T^2$, and $\Lambda_{\text{QCD}}^{N_f=4} = 234 \text{ MeV}$. This further improves the ultraviolet behaviour, via the inclusion of the one-loop \ln -suppression at $s_{\Omega_k} \gg \Lambda_{\text{QCD}}^2$, and through the Gaussian incorporates the intermediate-range enhancement observed in zero-temperature gluon DSE studies. The parameters D , m_t and ω can be fixed at $T = 0$ by requiring a good fit to a range of π - and K -meson properties. Herein we consider two phenomenologically equivalent parameter sets that differ only in the value of ω : $D = 0.78 \text{ GeV}^2$, $m_t = 0.5 \text{ GeV}$, and: ${}_1\omega = 0.6 m_t$, ${}_2\omega = 1.2 m_t$. At $T = 0$ in this model the quark mass function, $M(p^2) := B(p^2)/A(p^2)$, evolves according to the one-loop renormalisation group formula for $p^2 > 20 \Lambda_{\text{QCD}}^2$, and with ${}_1\omega$ a renormalisation point invariant current-quark mass of $\hat{m} = 6.6 \text{ MeV}$ yields $m_\pi = 140 \text{ MeV}$ while with ${}_2\omega$, $\hat{m} = 5.7 \text{ MeV}$ effects that.

An often used order parameter for dynamical chiral symmetry breaking is the quark condensate [16]:

$$-\langle \bar{q}q \rangle_\zeta := N_c \lim_{\bar{\Lambda} \rightarrow \infty} Z_4(\zeta, \bar{\Lambda}) \int_{l,p}^{\bar{\Lambda}} \text{tr}_D [S_0(p_{\omega_l})], \quad (20)$$

for each massless quark flavour. Here the subscript “0” denotes that the dressed-quark propagator is a chiral limit solution of Eq. (9), and $Z_4(\zeta, \bar{\Lambda})$ is the mass renormalisation constant: $Z_4(\zeta, \bar{\Lambda}) m_R(\zeta) = Z_2(\zeta, \bar{\Lambda}) m_{\text{bm}}(\bar{\Lambda})$. There are other, equivalent order parameters and from Eq. (20) it follows that one such is $\mathcal{X} := B_0(\vec{p}=0, \omega_0)$. This being a *bona fide* order parameter is particularly useful and important because it means that the lowest Matsubara frequency completely determines the character of the chiral phase transition.

The quark DSE obtained with \mathcal{D}_A is an algebraic equation. Its solution is therefore easy to analyse and either directly, via Eqs. (1) and (2), or using the susceptibilities, it is straightforward to establish [5] that this model has mean field critical exponents and to determine the critical temperature in Table I. The exponents are *unchanged* (e.g., $\mathcal{X}(0, h)^3$ is linear on $h \in [10^{-10}, 10^{-6}]$) and T_c reduced by $< 2\%$ ($T_c = \eta/(2\pi) = 0.15915\eta \rightarrow 0.15646\eta$) upon the inclusion of the higher-order $1/N_c$ -corrections to the dressed-quark-gluon vertex discussed explicitly in Refs. [21].

The critical behaviour of the \mathcal{D}_B -model is harder to explore because the solution of the quark DSE must be obtained numerically. The apparently simple interaction is actually more complicated to study than that defined with \mathcal{D}_C because the ultraviolet $\ln[s_{\Omega_k}]$ -suppression characteristic of asymptotically free theories is lacking. Therefore the renormalisation group properties are those of quenched QED, which has notable pathologies [22].

Chiral symmetry restoration in this model was studied in Ref. [5]. Herein, however, we report results obtained with an improved numerical procedure. The most significant change, implemented in order to make recovery of the $T = 0$ limit easier, was to place a fixed ultraviolet cutoff, ${}_4\Lambda$, on ${}_4q^2 := \vec{q}^2 + \omega_l^2$. This requires that a different three-momentum cutoff and grid be used for each Matsubara frequency. For small- T this corresponds closely to an $O(4)$ invariant cutoff procedure, and for all T it provides for an accurate discrete representation of the kernel. In the calculations we chose ${}_4\Lambda \sim 1.5\zeta \approx 14\text{ GeV}$, and for a given value of T the number of Matsubara modes required, l_{\max} , is determined from $\omega_{l_{\max}} \approx {}_4\Lambda$.

We employ two chiral order parameters in analysing chiral symmetry restoration

$$\mathcal{X} := B(\vec{p} = 0, \omega_0), \quad \mathcal{X}_C := \frac{B(\vec{p} = 0, \omega_0)}{C(\vec{p} = 0, \omega_0)}. \quad (21)$$

They should be equivalent and, as we will see, the onset of that equivalence is a good way to determine the h -domain on which Eqs. (5)-(7) are valid. Further, we have verified numerically that for $m = 0$ and $t \sim 0$: $f_\pi \propto \langle \bar{q}q \rangle \propto \mathcal{X}(t, 0)$; i.e., that these quantities are all *bona fide* order parameters. It thus follows from the pseudoscalar mass formula [16]: $f_\pi^2 m_\pi^2 = 2 m_R(\zeta) \langle \bar{q}q \rangle_\zeta^0$, that m_π diverges at the critical temperature [12],

$$m_\pi^2 \propto (-t)^{-\beta}, \quad t \rightarrow 0^-. \quad (22)$$

The critical temperature in Table I was obtained¹ using Eqs. (5). Its value is insensitive to whether t_{pc}^h or t_{pc}^t is used and also to the order parameter. $\chi_h^{\text{pc}}(h)$ and $\chi_t^{\text{pc}}(h)$, obtained using both order parameters, are depicted in Fig. 1 and, following Eqs. (6) and (7), the critical exponents are determined from these curves. In comparison with Ref. [5] it is obvious that there is significantly less error in the function values. This makes it possible to distinguish the nonzero curvature on the domain $-4.5 < \log_{10} h < -3$, whereas in Ref. [5] the error in the function values was such that the scaling relations, Eqs. (5)-(7), appeared to be valid on this domain. In fact, the scaling relations are not valid until

¹ In these calculations we corrected a numerical error in Refs. [5,12]: the Debye mass was a factor of three too large therein. That is the origin of the 15% increase in T_c , which also serves to illustrate the small influence of the perturbative Debye mass on the results.

$$\log_{10}(h/h_u) < -7, \quad (23)$$

$h_u := m_R/T_c$ is defined with the current-quark mass that gives $m_\pi = 140$ MeV in this model.

The scaling domain is most easily identified by defining a “local” critical exponent for each of the equivalent chiral order parameters:

$$z_i := -\frac{\ln \chi_i^{\text{pc}} - \ln \chi_{i+1}^{\text{pc}}}{\ln h_i - \ln h_{i+1}}, \quad (24)$$

where $(h_i, \chi_i^{\text{pc}})$ and $(h_{i+1}, \chi_{i+1}^{\text{pc}})$ are adjacent data pairs. $z_i(h)$ for the present model is depicted in Fig. 2. h lies in the scaling region when z_i is independent of the order parameter.

Anticipating the curvature, we fit the susceptibilities using the functional form

$$\log_{10} \chi^{\text{pc}} = \frac{1 + a_0 \varsigma + a_1 \varsigma^2}{a_2 + a_3 \varsigma}, \quad (25)$$

$\varsigma = \log_{10} h$, over the smallest five values of ς , and the critical exponent is then: $z = a_1/a_3$. We estimate the error by systematically eliminating one value of ς at a time from the fit and observing the change in z . In this way we obtain the values of z_h and z_t in Table I.

A linear fit on $\varsigma \in [-4.5, -3]$ yields: $z_h = 0.79$, $z_t = 0.40$, which may be compared with the values in Ref. [5]: $z_h = 0.77 \pm 0.02$, $z_t = 0.28 \pm 0.04$. Comparing the values of z_h highlights the error introduced by incorrectly identifying the scaling region. The comparison between the values of z_t indicates our significant improvement of the numerical method.

The study of chiral symmetry restoration in the model obtained with \mathcal{D}_C is straightforward, with the additional $\ln[s_{\Omega_k}]$ -suppression in the ultraviolet making the numerical analysis much simpler. The critical temperature for each parameter set is presented in Table I, and on this ω -domain: $T_c(\omega) \approx 88 + 107 \omega$. $\chi_h^{\text{pc}}(h)$ and $\chi_t^{\text{pc}}(h)$ for the smallest value of ω , which corresponds to the parameter set in Ref. [16], are depicted in Fig. 3. The results for the larger value of ω are similar. In these cases the scaling relations are only valid for

$$_1\omega: \log_{10}(h/h_u) < -5, \quad _2\omega: \log_{10}(h/h_u) < -6. \quad (26)$$

It is clear from Table I that each of these models is mean field in nature. In hindsight that may appear unsurprising because the long-range part of the interaction is identical in each case and the correlation length diverges as $t \rightarrow 0$. However, the models differ in detail by the manner in which the interactions approach their long-range limits, and our numerical demonstration of their equivalence required a very careful analysis and extremely small values of the current-quark mass, Eqs. (23) and (26).

This last observation is also likely to be true in QCD; i.e., while the critical temperature is relatively easy to determine, very small current-quark masses may be necessary to accurately calculate the critical exponents from the chiral and thermal susceptibilities. If that is the case, the calculation of these exponents via numerical simulations of lattice-QCD will not be feasible. The discrepancies described in Ref. [7] could be a signal of this.

The class of models we have considered explicitly preserves the chiral Ward-Takahashi identity and can describe the long-wavelength dynamics of QCD very well [8,16]. Importantly, it describes that dynamics in terms of mesons that are quark-antiquark *composites*. The characteristic feature of this class is the behaviour of the long-range, confining interaction. It provides an additive, algebraic driving term in the quark DSE that is proportional

to the dressed-quark propagator, which means that boson Matsubara-frequency zero-modes do not influence the critical behaviour determined from the gap equation.

The class of Coulomb gauge models considered in Ref. [9] also describes mesons as composite particles and it too exhibits mean field critical exponents. The long-range part of the interaction in that class of models corresponds directly to the regularised Fourier amplitude of a linearly rising potential. Hence it is not equivalent to ours in any simple way, except insofar as zero modes do not influence the gap equation.

We have also considered the class of models obtained with [10]: $\mathcal{D}(p_{\omega_k} - q_{\omega_l}; m_g) \propto g(|\vec{p}|) g(|\vec{q}|)$, where $g(|\vec{p}|)$ is a non-increasing function of its argument. This class includes the non-confining NJL-model, which is obtained with $g(|\vec{p}|) = \theta(1 - |\vec{p}|/\Lambda)$, where Λ is a cutoff parameter, and can also provide a good description of long-wavelength pion dynamics. The models in this class describe mesons as composites and exhibit an explicit fermion substructure, and those we have explored have mean field critical exponents. The same is true of separable models with $g_i = g_i(\omega_k^2 + \vec{p}^2)$ [23].

The quark DSE is the QCD gap equation and the many equivalent chiral order parameters are directly related to properties of its solution. We have observed that four classes of models exhibit the same (mean field) critical exponents. The classes are distinguished by the qualitatively distinct type of interaction they employ in the gap equation but similar in using the rainbow truncation. Only in our simplest confining model did we consider the effect of $1/N_c$ -corrections to the quark-gluon vertex, and in that case the critical exponents were unchanged. Incipient in these results is the suggestion that mean field exponents are a feature of the essential fermion substructure in the gap equation, and not of the interaction.² If that is correct then chiral symmetry restoration at finite- T in QCD is a mean field transition. This hypothesis is supported by chiral random matrix models of QCD, whose formulation relies only on the symmetries of the Dirac operator and which also yield [24] mean field exponents. It can likely only be false if $1/N_c$ -corrections to the vertex are large in the vicinity of the transition, in which case their null effect in our simplest model will have been misleading.

ACKNOWLEDGMENTS

AH acknowledges the hospitality of the Physics Division at Argonne National Laboratory, and PM and CDR that of the Physics Department at the University of Rostock during visits facilitating this work, which was supported in part by: Deutscher Akademischer Austauschdienst; the US Department of Energy, Nuclear Physics Division, under contract number W-31-109-ENG-38; the National Science Foundation under grant no. INT-9603385; and benefited from the resources of the National Energy Research Scientific Computing Center.

² Our discussion is independent of the number of light quark flavours insofar as that number does not suppress the interaction strength to such an extent that chiral symmetry is not dynamically broken at $T = 0$.

REFERENCES

- [1] R. Pisarski and F. Wilczek, Phys. Rev. D **29**, 338 (1984); F. Wilczek, Int. J. Mod. Phys. A **7**, 3911 (1992).
- [2] G. Baker, B. Nickel and D. Meiron, Phys. Rev. B **17**, 1365 (1978); and “Compilation of 2-pt. and 4-pt. graphs for continuous spin models”, University of Guelph report (1977), unpublished.
- [3] A. Kocić and J. Kogut, Phys. Rev. Lett. **74**, 3109 (1995); Nucl. Phys. B **455**, 229 (1995).
- [4] J. B. Kogut, M. A. Stephanov and C. G. Strouthos, “Critical region of the finite temperature chiral transition”, hep-lat/9805023
- [5] D. Blaschke, A. Höll, C. D. Roberts and S. Schmidt, Phys. Rev. C **58**, 1758 (1998).
- [6] F. Karsch and E. Laermann, Phys. Rev. D **50**, 6954 (1994).
- [7] E. Laermann, Nucl. Phys. B **63** (Proc. Suppl.), 114 (1998).
- [8] P. C. Tandy, Prog. Part. Nucl. Phys. **39**, 117 (1997); C. D. Roberts, “Nonperturbative QCD with Modern Tools”, nucl-th/9807026.
- [9] R. Alkofer, P. A. Amundsen and K. Langfeld, Z. Phys. C **42**, 199 (1989).
- [10] S. Schmidt, D. Blaschke and Yu. L. Kalinovsky, Phys. Rev. C **50**, 435 (1994).
- [11] P. Maris, C. D. Roberts and S. Schmidt, Phys. Rev. C **57**, R2821 (1998); D. Blaschke, C.D. Roberts and S. Schmidt, Phys. Lett. B **425**, 232 (1998).
- [12] A. Bender, D. Blaschke, Yu. Kalinovsky and C.D. Roberts, Phys. Rev. Lett. **77**, 3724 (1996).
- [13] A. Bender, *et al.*, Phys. Lett. B **431**, 263 (1998) pp. 263-269
- [14] A. Bashir, A. Kizilersu and M.R. Pennington, Phys. Rev. D **57**, 1242 (1998); and references therein.
- [15] F. T. Hawes, P. Maris and C. D. Roberts, “Infrared Behaviour of Propagators and Vertices”, nucl-th/9807056, Phys. Lett. B., in press.
- [16] P. Maris and C. D. Roberts, Phys. Rev. C **56**, 3369 (1997).
- [17] H.J. Munczek and A.M. Nemirovsky, Phys. Rev. D **28**, 3081 (1983).
- [18] M. Baker, J. S. Ball and F. Zachariasen, Nucl. Phys. B **186** 531 (1981); *ibid* 560; D. Atkinson, P. W. Johnson, W. J. Schoenmaker and H. A. Slim, Nuovo Cimento A **77**, Series 11 (1983) 197.
- [19] N. Brown and M. R. Pennington, Phys. Rev. D **39**, 2723 (1989); M. R. Pennington, “Calculating hadronic properties in strong QCD”, hep-ph/9611242.
- [20] M.R. Frank and C.D. Roberts, Phys. Rev. C **53**, 390 (1996).
- [21] A. Bender, C. D. Roberts and L. v. Smekal, Phys. Lett. B **380** (1996) 7; C. D. Roberts, in *Quark Confinement and the Hadron Spectrum II*, edited by N. Brambilla and G. M. Prosperi (World Scientific, Singapore, 1997), pp. 224-230.
- [22] F. T. Hawes, T. Sizer and A. G. Williams, Phys. Rev. D **55**, 3866 (1997).
- [23] P. C. Tandy, private communication.
- [24] A. D. Jackson and J. J. M. Verbaarschot, Phys. Rev. D **53**, 7223 (1996).

TABLES

	A	B	$C_{1\omega}$	$C_{2\omega}$
$T_c(MeV)$	169	174	120	152
z_h	0.666	0.67 ± 0.01	0.667 ± 0.001	0.669 ± 0.005
z_t	0.335	0.33 ± 0.02	0.333 ± 0.001	0.33 ± 0.01

TABLE I. Critical temperature for chiral symmetry restoration and critical exponents characterising the second-order transition in the four exemplary models. Mean field critical exponents are: $z_h = 2/3$, $z_t = 1/3$.

FIGURES

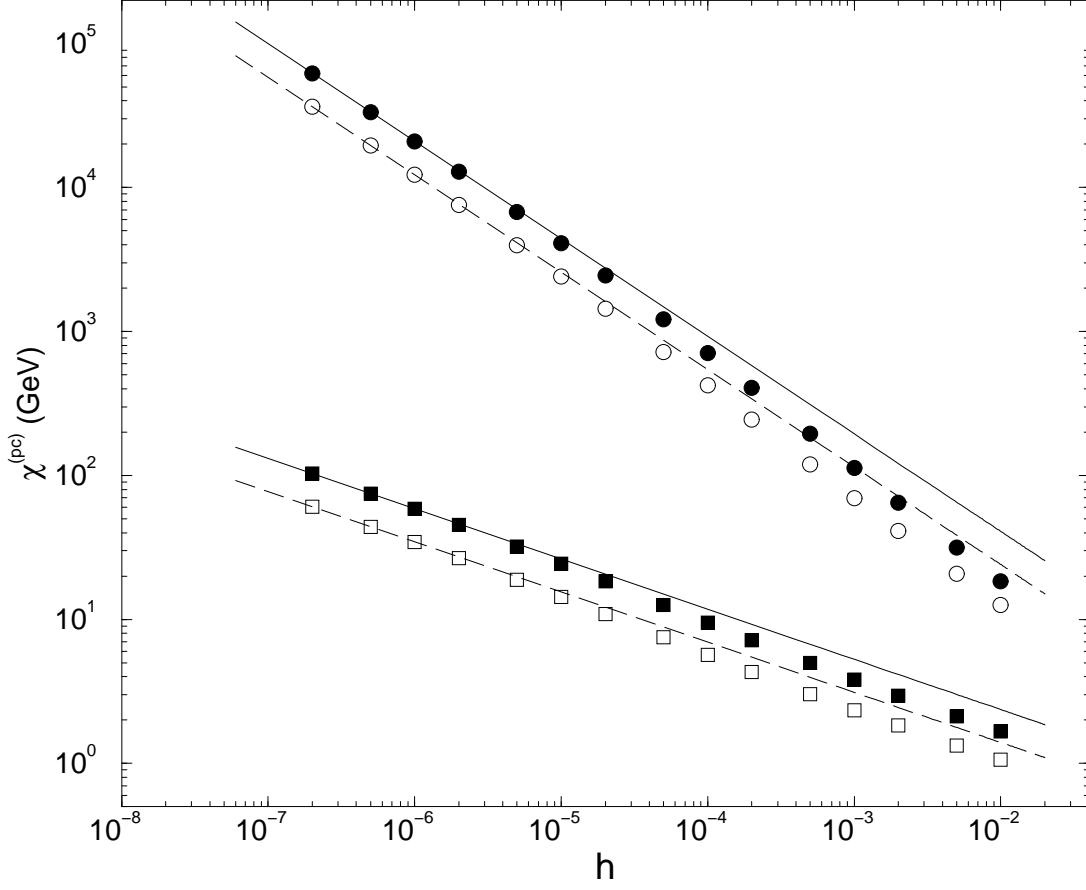


FIG. 1. $\chi_h^{\text{pc}}(h)$ (circles) and $\chi_t^{\text{pc}}(h)$ (squares) calculated in the model defined via \mathcal{D}_B in Eq. (18): filled symbols - \mathcal{A} , open symbols - \mathcal{A}_C . The slope of the straight lines is given in Table I and they are drawn through the two smallest h -values.

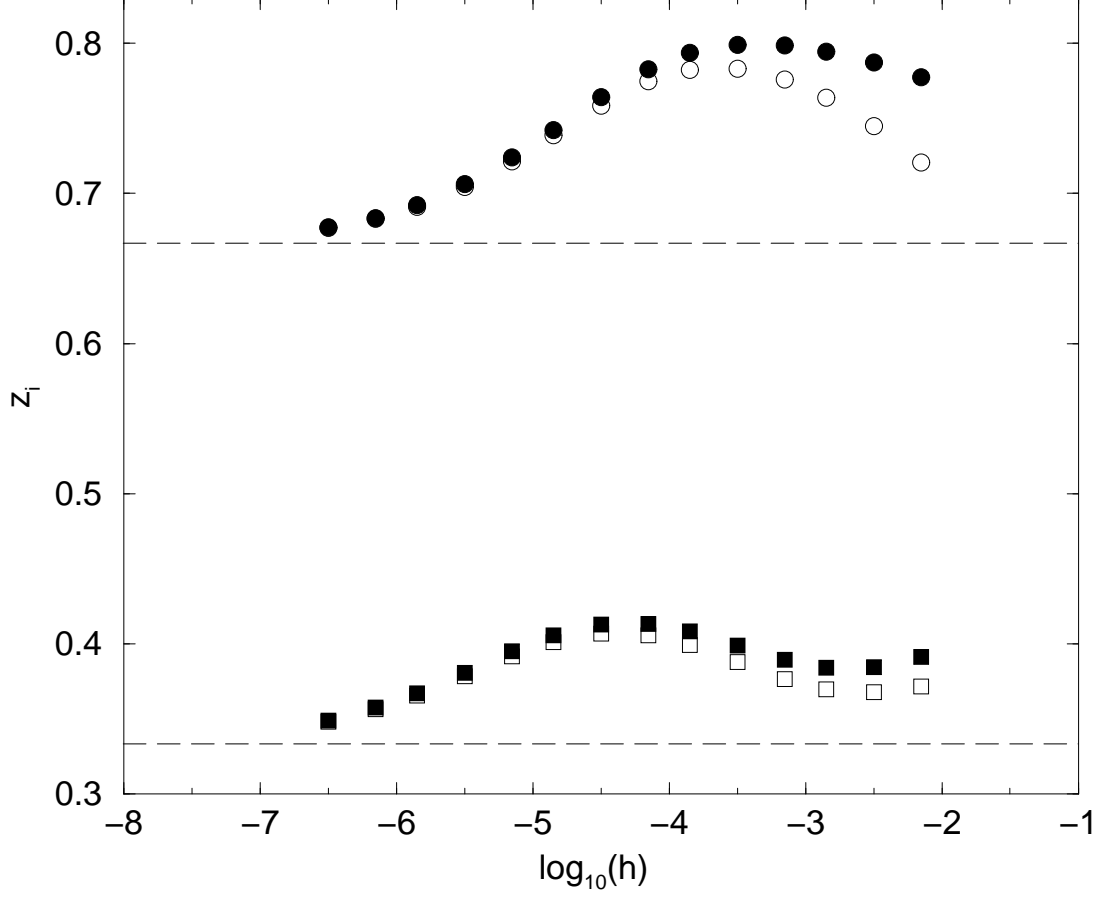


FIG. 2. z_i^h (circles) and z_i^t (squares) from Eq. (24) in the model defined via \mathcal{D}_B in Eq. (18): filled symbols - \mathcal{X} , open symbols - \mathcal{X}_C . The dashed lines are the mean field values: $z_h = 2/3$, $z_t = 1/3$.

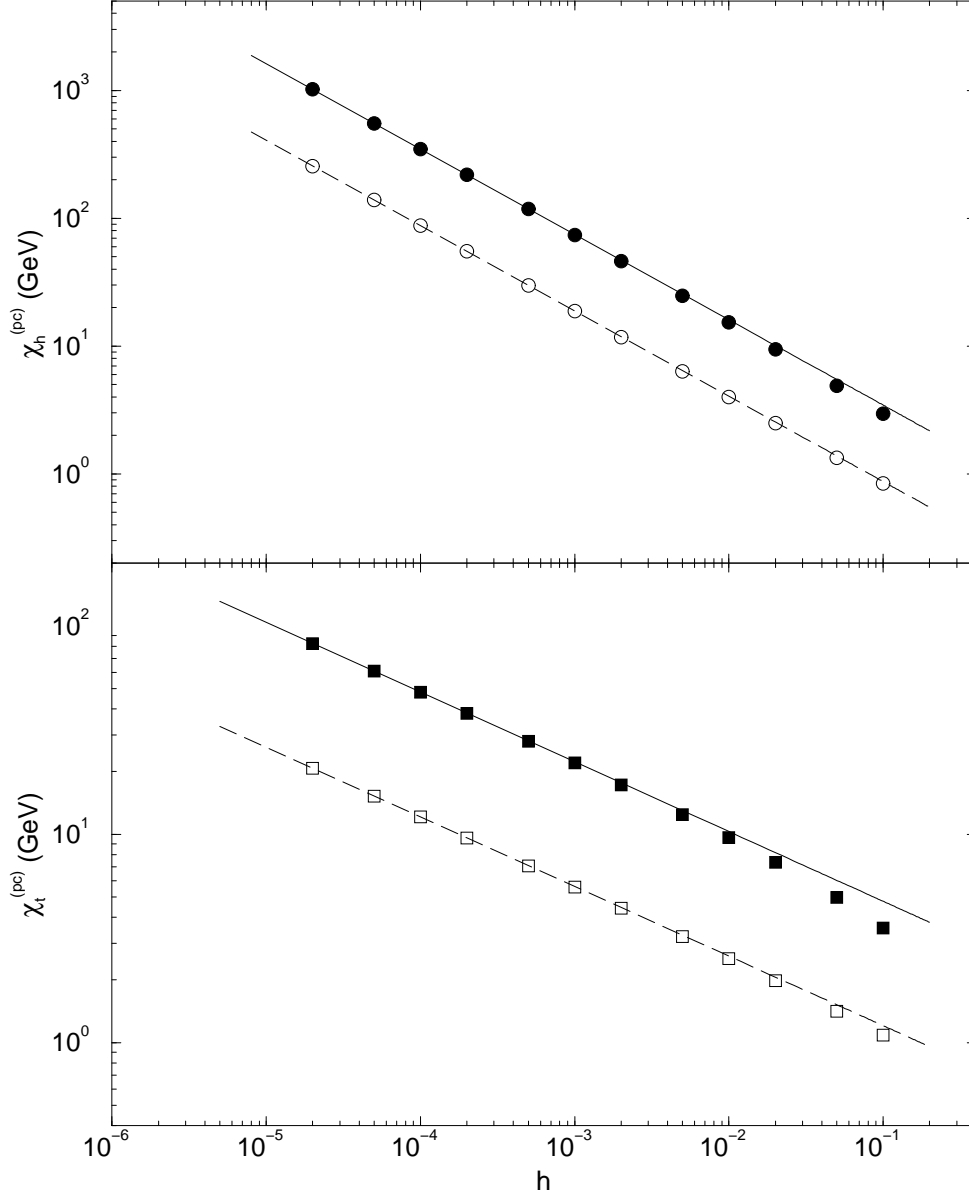


FIG. 3. $\chi_h^{\text{pc}}(h)$ (circles) and $\chi_t^{\text{pc}}(h)$ (squares) calculated in the model defined via \mathcal{D}_C in Eq. (19) with $\omega = 0.6 m_t$: filled symbols - \mathcal{A} , open symbols - \mathcal{A}_C . The slope of the straight lines is given in Table I and they are drawn through the two smallest h -values.